Tutorial 7

1. **Cooperative games**

Exercise 1. *Consider bimatrix game*

$$(A, B) = \begin{pmatrix} (a, 2) & (3, 0) \\ (2, 0) & (2, 2) \end{pmatrix},$$

where $a > 2$ is arbitrary. Find the maximin values of the two players and the arbitrary pair as functions of a.

Solution. Since

$$B = B^T = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix},$$

the maximin value for Player II is $v_{B^T} = 1$. To find v_A, note that for $x \in [0, 1]$,

$$(x, 1-x)A = (x, 1-x) \begin{pmatrix} a & 3 \\ 2 & 2 \end{pmatrix} = ((a-2)x + 2, x + 2).$$

We need to consider three cases: $a = 3$; $a > 3$; and $2 < a < 3$.

If $a = 3$, A has a saddle point 3, hence $v_A = 3$. The status quo point is $(v_A, v_{B^T}) = (3, 1)$. We draw the cooperative region as in Figure 3. In this case, the bargaining set is the singleton $\{(3, 2)\}$. Hence the arbitration pair is $(3, 2)$.

If $a > 3$, A still has saddle point 3, hence $v_A = 3$. The cooperative region is shown in Figure 4. Now the bargaining set is the singleton $\{(a, 2)\}$. Hence the arbitration pair is $(a, 2)$.

If $2 < a < 3$, A has saddle point a, then $v_A = a$. The status quo point is $(a, 1)$. We draw the cooperative region as in Figure 5. The bargaining set

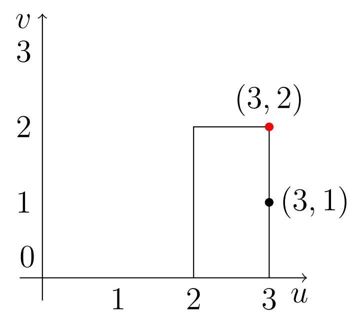

Figure 1

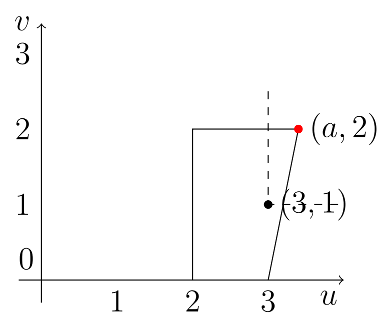

Figure 2

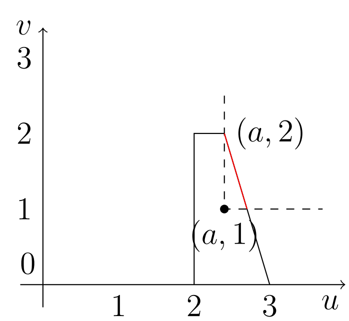

Figure 3

is the line segment joining $(a, 2)$ and $(\frac{a+3}{2}, 1)$. To find the arbitration pair, consider

$$g(u, v) = (u - a)(v - 1).$$

The line joining $(a, 2)$ and $(3, 0)$ is $v = \frac{2}{a-3}(u-3)$. Hence in the bargaining set, we have

$$\begin{aligned} g(u, v) &= (u - a)(\frac{2}{a-3}(u-3) - 1) \\ &= \frac{2}{a-3}u^2 - \frac{3(a+1)}{a-3}u + \frac{a(a+3)}{a-3}. \end{aligned}$$

It is easy to see g attains its maximum at $u = \frac{3(a+1)}{4}$, $v = \frac{3}{2}$. Hence the arbitration pair in this case is $(\frac{3(a+1)}{4}, \frac{3}{2})$.

To sum up, we have: if $a \geq 3$, then $v_A = 3, v_{B^T} = 1$, the arbitration pair is $(a, 2)$; if $2 < a < 3$, then $v_A = a, v_{B^T} = 1$, the arbitration pair is $(\frac{3(a+1)}{4}, \frac{3}{2})$.

Exercise 2. *Consider the two-person game with bimatrix*

$$(A, B) = \begin{pmatrix} (5, 0) & (-3, 1) & (-2, 0) \\ (2, 4) & (2, 3) & (4, -2) \end{pmatrix}.$$

(i) Find v_A and v_{B^T}.

(ii) Sketch the cooperative region of the game.

(iii) Using $(\mu, \nu) = (v_A, v_{B^T})$ as the status quo point, find the arbitration pair of the game.

Solution. (i) We have

$$A = \begin{pmatrix} 5 & -3 & -2 \\ 2 & 2 & 4 \end{pmatrix}, B = \begin{pmatrix} 0 & 1 & 1 \\ 4 & 3 & -2 \end{pmatrix}, B^T = \begin{pmatrix} 0 & 4 \\ 1 & 3 \\ 0 & -2 \end{pmatrix}.$$

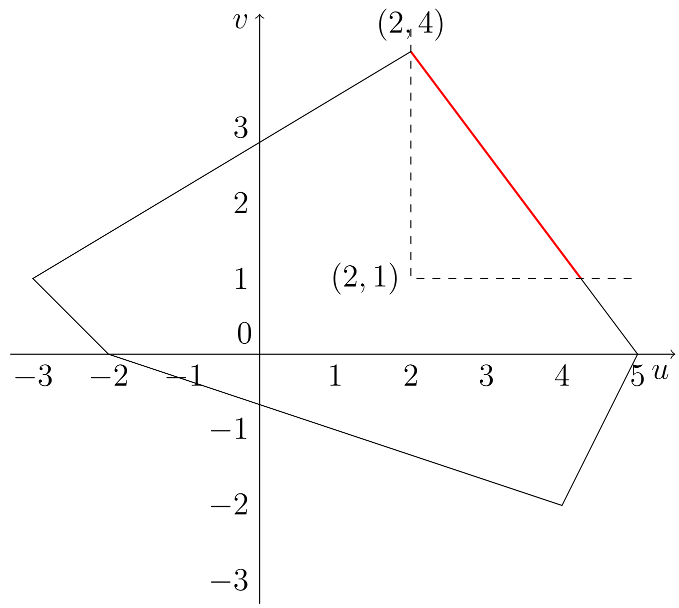

Figure 4

Note that A has saddle point 2, B has saddle point 1. Hence $v_A = 2, v_{B^T} = 1$.

(ii) The cooperative region is shown in Figure 1.

(iii) The line joining $(2, 4)$ and $(5, 0)$ is given by

$$v = \frac{0-4}{5-2}(u-5) = -\frac{4}{3}(u-5).$$

Let $v = 1$, we have $u = \frac{17}{4}$. Hence the bargaining set is the line segment joining points $(2, 4)$ and $(\frac{17}{4}, 1)$. Let $g(u, v) = (u-2)(v-1)$. Then on the bargaining set, we have

$$g(u, v) = (u-2)(-\frac{4}{3}(u-5)-1) = -\frac{4}{3}u^2 + \frac{25}{3}u - \frac{34}{3}.$$

It is easy to see that g attains its maximum at $u = \frac{25}{8}$. In this case, $v = -\frac{4}{3}(\frac{25}{8} - 5) = \frac{5}{2}$. Hence the arbitration pair is $(\frac{25}{8}, \frac{5}{2})$.